sgα-Closed sets in Topological Spaces

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Abstract - In this paper, a new set called sgα-closed set is introduced. Also, its properties were studied.

***Keywords:** sgα-closed sets and sgα-open sets.*

I. INTRODUCTION

N. Levine [7] introduced generalized closed sets (briefly g-closed set) in 1970. N. Levine [12] introduced the concepts of semi-open sets in 1963. Bhattacharya and Lahiri [3] introduced and investigated semi-generalized closed (briefly sg- closed) sets in 1987. Arya and Nour [2] defined generalized semi-closed (briefly gs-closed) sets for obtaining some characterization of s-normal spaces in 1990. O.Njastad in 1965 defined α-open sets [12]. In this paper, a new set called semi-generalized α-closed sets (briefly sgα-closed) is introduced their proper- ties were studied.

Throughout the paper X and Y denote the topological spaces (X, τ) and (Y, σ) respectively and on which no separation axioms are assumed unless otherwise explicitly stated.

II. PRELIMINARIES

A subset A of a topological space X is said to be open if A ∈ τ A subset A of a topological space X is said to be closed if the set X − A is open. The interior of a subset A of a topological space X is defined as the union of all open sets contained in A. It is denoted by int(A). The closure of a subset A of a topological space X is defined as the intersection of all closed sets containing A. It is denoted by cl(A).

Definitions 2.1.

1. A subset A of a space (X, τ) is said to be semi open [6] if A ⊆ cl (int (A)) and semi closed if int (cl (A)) ⊆ A.

2. A subset A of a space (X, τ) is said to be α-open [12] if A ⊆ int (cl (int (A))) and α-closed if cl (int (cl (A))) ⊆ A.

3. A subset A of a space (X, τ) is said to be β-open or semi pre- open [1] if A ⊆ cl (int (cl (A))) and β-closed or semi pre-closed if int (cl (int (A))) ⊆ A.

4. A subset A of a space (X, τ) is said to be pre-open [11] if A ⊆ int (cl (A))

and pre-closed if cl (int (A)) ⊆ A.

The complement of a semi-open (resp.pre-open, α-open, β-open) set is called semi-closed (resp.pre-closed, α-closed, β-closed). The intersec- tion of all semi-closed (resp.pre-closed, α-closed, β-closed) sets containing A is called the semi-closure (resp.pre-closure, α-closure, β-closure) of A and is denoted by scl(A)(resp. pcl(A), α-cl(A), β-cl(A)) . The union of all semi-open (resp.pre-open, α-open, β-open) sets contained in A is called the semi-interior(resp.pre-interior, α-interior, β-interior) of A and is de- noted by sint(A)(resp. pint(A), α-int(A), β-int(A)). The family of all semi- open (resp.pre-open, α-open, β-open)sets is denoted by SO(X)(resp. P O(X), α − O(X), β − O(X)). The family of all semi-closed (resp.pre-closed, α-closed, β-closed)sets is denoted by SC l(X) (resp. P C l(X), α-C l(X), β-C l(X)). Definitions 2.2.

1. A subset A of a space (X, τ) is called generalized-closed set [7] (briefly g-closed) if cl (A) ⊆ U , whenever A ⊆ U and U is open in (X, τ).The complement of a g-closed set is called g-open set.

2. A subset A of a space (X, τ) is called generalized semi-closed set [12] (briefly gs-closed set) if scl (A) ⊆ U, whenever A ⊆ U and U is open in (X, τ).

3. A subset A of a space (X, τ) is called semi-generalized closed set [3] (briefly sg-closed set) if scl (A) ⊆ U, whenever A ⊆ U and U is semi-open in (X, τ).

4. A subset A of a space (X, τ) is called α generalized-closed set [9] (briefly αg-closed) if α (cl (A)) ⊆ U, whenever A ⊆ U and U is open in (X, τ).

5. A subset A of a space (X, τ) is called generalized α-closed set [8] (briefly gα-closed) if α (cl (A)) ⊆ U, whenever A ⊆ U and U is α-open in (X, τ).

6. A subset A of a space (X, τ) is called generalized pre-closed set [10] (briefly gp-closed) if pcl (A) ⊆ U, whenever A ⊆ U and U is open in (X, τ).

7. A subset A of a space (X, τ) is called generalized semi-pre closed set [4] (briefly gsp-closed) if spcl (A) ⊆ U, whenever A ⊆ U and U is open in (X, τ).

III. sgα-CLOSED SETS IN TOPOLOGICAL SPACES

In this section the notion of a new class of sets called sgα-closed sets in topo- logical spaces is introduced and their properties were studied.

Definition 3.1 A subset A of space (X, τ) is called sgα-closed if scl (A) ⊆

U, whenever A ⊆ U and U is α-open in X.

The family of all sgα-closed subsets of the space X is denoted by SGαC (X).

Definition 3.2 The intersection of all sgα-closed sets containing a set A is called sgα-closure of A and is denoted by sgα-cl(A).

A set A is sgα-closed set if and only if sgα Cl (A) = A.

Definition 3.3 A subset A in X is called sgα-open in X if A^c is sgα- closed in X.

The family of a sgα-open sets is denoted by SGαO(X).

Definition 3.4 The union of all sgα-open sets containing a set A is called sgα-interior of A and is denoted by sgα-Int(A).

A set A is sgα-open set if and only if sgα Int (A) = A.

Theorem 3.5 Every closed set is a sgα-closed set.

Proof: Let A be a closed set and U be any α-open set containing A. Since A is closed, cl (A) = A. For every subset A of X, scl (A) ⊆ cl (A) = A ⊂ U and so we have scl (A) ⊆ U.Hence A is sgα-closed.

Remark 3.6 The converse of the above theorem need not be true as seen from the following example.

Example 3.7 Let X = {a, b, c} with topology τ = {X, φ, {b, c}}. Then

A = {a, b} is sgα-closed but not a closed set of (X, τ).

Theorem 3.8 Every gα closed set is a sgα-closed set.

Proof: Proof follows from the definition obviously.

Remark 3.9 The converse of the above theorem need not be true as seen from the following example.

Example 3.10 Let X = {a, b, c} with topology τ = {X, φ, {a}, {c}, {a, c} {b, c}}. Then A = {b} is sgα-closed but not gα closed set of (X, τ).

Theorem 3.11 Every sgα closed set is a sg-closed set.

Proof: The proof follows from the definition and the fact that every semi-open set is α-open.

Remark 3.12 The converse of the above theorem need not be true as seen from the following example.

Example 3.13 Let X = {a, b, c} with topology τ = {X, φ, {c}, {a, c}}. Then A = {a} is sg-closed but not sgα closed set of (X, τ).

Theorem 3.14 Every sgα closed set is a gs-closed set.

Proof: The proof follows from the definition and the fact that every open set is α-open.

Remark 3.15 The converse of the above theorem need not be true as seen from the following example.

Example 3.16 Let X = {a, b, c} with topology τ = {X, φ, {c}, {a, c}}. Then A = {b, c} is gs-closed but not sgα closed set of (X, τ).

Theorem 3.17 Every sgα closed set is a gsp-closed set.

Proof: Let A be a sgα-closed set.Let A ⊆ U and U be open.Then A ⊆ U and U is α-open and scl (A) ⊆ U. Since every open set is α-open. A is sgα-closed.Then spcl (A) ⊆ scl (A) ⊆ U. Hence A is gsp-closed.

Remark 3.18 The converse of the above theorem need not be true as seen from the following example.

Example 3.19 Let X = {a, b, c} with topology τ = {X, φ, {a, b}}. Then A = {a} is gsp-closed but not sgα closed set of (X, τ).

Theorem 3.20 The union of two sgα-closed subsets of X is also sgα-closed subset of X.

Proof: Assume that A and B are sgα-closed set in X. Let U be α-open in X such that A ∪ B ⊂ U. Then A ⊂ U and B ⊂ U. Since A and B are sgα-closed, scl (A) ⊂ U and scl (B) ⊂ U. Hence scl (A ∪ B) = (scl (A)) ∪ (scl (B)) ⊂ U. That is scl (A ∪ B) ⊂ U. Therefore A ∪ B is sgα-closed set in X.

Remark 3.21 The intersection of two sgα-closed sets in X is generally not sgα-closed set in X.

Example 3.22 Let X = {a, b, c, d, e} with topology τ = {X, φ, {a}, {d}, {e}, {a, d}, {a, e}, {d, e}, {a, d, e}}. If A = {a, b, c} and B = {a, d, e}, then A and B are sgα-closed sets in X, but A ∩ B = {a} is not a sgα-closed set of X.

Theorem 3.23 If a subset A of X is sgα-closed set in X. Then scl (A) |A does not contain any nonempty α-open set in X.

Proof: Suppose that A is sgα-closed set in X. We prove the result by con- tradiction. Let U be a α-open set such that scl (A) |A ⊃ U and U = φ. Now U ⊂ scl (A) |A. Therefore U ⊂ X |A which implies A ⊂ X |U. Since U is α-open set, X |U is also α-open in X. Since A is sgα-closed set in X, by definition we have scl (A) ⊂ X |U. So U ⊂ X |scl (A). Also U ⊂ scl (A). Therefore U ⊂ (scl (A) ∪ (X |scl((A))) = φ. This shows that, U = φ which is contradiction. Hence scl (A) |A does not contains any nonempty α-open set in X.

Remark 3.24 The converse of the above theorem need not be true seen from following example.

Example 3.25 If scl (A) |A contains no nonempty sgα-open subset in X, then A need not be sgα-closed set. Let X = {a, b, c, d, e} with topology τ = {X, φ, {a}, {d}, {e}, {a, d}, {a, e}, {d, e}, {a, d, e}} and A = {a, b}. Then scl (A) |A = {a, b, c} | {a, b} = {c} does not contain nonempty α-open set in X, but A is not a sgα-closed set in X.

Corollary 3.26 If a subset A of X is sgα-closed set in X then scl (A) |A does not contain any open set in X but not conversely.

Proof: Follows from theorem 3.23 and the fact that every open set is α-open.

Corollary 3.27 If a subset A of X is sgα-closed set in X then scl (A) |A does not contain any non empty closed set in X but not conversely.

Proof: Follows from theorem 3.23 and the fact that every open set is α-open.

Theorem 3.28 For an element x ∈ X, the set X| {x} is sgα-closed or α-open.

Proof: Suppose X| {x} is not α-open set. Then X is the only α-open set containing X | {x}. This implies sclX | {x} ⊂ X. Hence X | {x} is sgα-closed set in X.

Theorem 3.29 If A is open and sgα-closed then A is closed and hence α-clopen.

Proof: Suppose A is open and sgα-closed. As every open is α-open and A ⊂ A, we have scl (A) ⊂ A. Also A ⊂ scl (A). Therefore scl (A) = A. That is A is α-closed. Since A is open, A is α-open. Now cl (int (A)) = cl (A). Therefore A is closed and α-clopen.

Theorem 3.30 If A is sgα-closed subset of X such that A ⊂ B ⊂ scl (A). Then B is sgα-closed set in X.

Proof: If A is sgα-closed subset of X such that A ⊂ B ⊂ scl (A). Let U be a α-open set of X such that B ⊂ U. Then A ⊂ U. Since A is a sgα-closed we have scl (A) ⊂ U. Now scl (B) ⊂ scl (scl (A)) = scl (A) ⊂ U. Therefore B is sgα-closed set in X.

Theorem 3.31 If A is sgα-closed and A ⊂ B ⊂ scl (A), then B is sgα- closed.

Proof: Let A be sgα-closed and B ⊂ U, where U is α-open. Then A ⊂ B implies A ⊂ U. Since A is sgα-closed, scl (A) ⊂ U. B ⊂ scl (A) implies scl (B) ⊂ scl (A). Therefore scl (B) ⊂ U and hence B is sgα-closed.

Remark 3.32 The converse of the theorem 3.31 need not be true in general as seen from following example.

Example 3.33 Let X = {a, b, c, d, e} with topology τ = {X, φ, {a}, {d}, {e}, {a, d}, {a, e}, {d, e}, {a, d, e}}. A = {b} and B = {b, c}. Then A and B are sgα-closed sets in (X, τ),but A ⊂ B is not subset in scl (A).

Theorem 3.34 Let A be a sgα-closed in (X, τ). Then A is α-closed if and only if scl (A) |A is a α-open.

Proof: Suppose A is a α-closed in X. Then scl (A) = A and so scl (A) |A = φ, which is α-open in X. Conversely, suppose scl (A) |A is α-open set in X. Since A is sgα-closed by theorem 3.23.scl (A) |A does not contain any non empty α- open in X. Then scl (A) |A = φ, hence A is α-closed set in X.

Theorem 3.35 If a subset A of topological space X is both α-open and sgα-closed, then it is α-closed.

Proof: Suppose a subset A of topological space X is both α-open and sgα- closed. Let A ⊂ U with U is α-open in X. Now A ⊃ int (cl (int (A))), as A is α-open. That is scl (A) ⊂ A ⊂ U. Thus A is sgα-closed.

Corollary 3.36 Let A be α-open and sgα-closed subset in X. Suppose that F is α-closed set in X. Then A ∩ F is an sgα-closed set in X.

Proof: Let A be a α-open and sgα-closed subset in X and F be closed. By theorem 3.14, A is α-closed. So A ∩ F is a α-closed and hence A ∩ F is sgα-closed.

Theorem 3.37 In a topological space X, if SαO(X) = {X, φ}, then every subset of X is a sgα-closed set.

Proof: Let X be a topological space and SαO(X) = {X, φ}. Let A be any subset of X. Suppose A = φ. Then φ is sgα-closed set in X. Suppose A = φ. Then X is the only α-open set containing A and so scl (A) ⊂ U. Hence A is sgα-closed set in X.

Remark 3.38 The converse of the above theorem need not be true in gen- eral as seen from the following example.

Example 3.39 Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \varphi, \{a, b\}, \{c, d\}\}$. Then every subset of (X, τ) is sgα-closed set in X, But $S\alpha O(X, \tau) = \{X, \varphi, \{a, b\}, \{c, d\}\}$.

Theorem 3.40 In a topological space X, $S\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$ if and only if every subset of X is a sgα-closed set.

Proof: Suppose that $S\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. Let A be any sub- set of X such that $A \subset U$, where U is a α-open. Then $U \in S\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$. That is $U \in \{F \subset X : F^c \in \tau\}$. Thus U is a α-closed set, then scl (U) = U. Also scl (A) ⊂ scl (U) = U. Hence A is sgα- closed set in X. Conversely, suppose that every subset of (X, τ) is sgα- closed. Let $U \in S\alpha O(X, \tau)$. Since $U \subset U$ and U is sgα-closed, we have scl (U) ⊂ U. Thus scl (U) = U and $U \in \{F \subset X : F^c \in \tau\}$. Therefore $S\alpha O(X, \tau) \subset \{F \subset X : F^c \in \tau\}$.

Definition 3.41 The intersection of all semi generalized α-open subsets of (X, τ) containing A is called the semi generalized α-kernal of A and is denoted by sg − rαker (A).

Lemma 3.42 Let X be a topological space and A be a subset of X. If A is a α-open in X, then sg − rαker (A) = A but not conversely.

Proof: Follows from definition 3.41.

Lemma 3.43 For any subset A of X ,sg − rαker (A) ⊂ sg − rαker (A).

Proof: Follows from implication SαO(X) ⊂ αO(X).

Lemma 3.44 For any subset A of X, A ⊂ sg − rαker (A).

Proof: Follows from definition 3.41.

Theorem 3.45 A subset A of (X, τ) is sgα-closed if and only if scl (A) ⊂ sg − rαker (A).

Proof: Suppose that A is sgα-closed. Then scl (A) ⊂ U, whenever A ⊂ and U is α-open. Let x ∈ scl (A). Suppose x ∉ sg − rαker (A); then there is a α-open set U containing A such that x is not in U. Since A is sgα- closed, scl (A) ⊂ U. We have x not in scl (A), which is a contradiction. Hence x ∈ sg − rαker (A) and so scl (A) ⊂ sg − rαker (A). Conversely, Let scl (A) ⊂ sg − rαker (A). If U is any α-open set containing A, then sg − rαker (A) ⊂ U. That is scl (A) ⊂ sg − rαker (A) ⊂ U. Therefore, A is sgα-closed in X.

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