Connected Eccentric Domination Number in Graphs

R. Jahir Hussain [1], A. Fathima Begam[2]
[1,2]PG and Research Department of Mathematics, Jamal Mohamed College (Autono)
Tiruchirappalli-620020, Tamilnadu, India

Abstract

A set $D \subseteq V(G)$ is a connected eccentric dominating set if D is an eccentric dominating set and also the induced subgraph <D> is connected. The cardinality of minimum connected eccentric dominating set is known as the connected eccentric domination number and is denoted by γ_{ce} (G). In this paper we present some bounds on the connected eccentric domination number and exact values of some particular graphs.

Keywords

Connected eccentric dominating set, minimum connected eccentric dominating set, minimal connected eccentric dominating set, connected eccentric domination number, eccentric dominating set, eccentric domination number.

1. Introduction

The first paper in Graph Theory was written by Euler in 1736 when he settled the famous unsolved problem of his day, known as the Konigsberg Bridge problem. This paper, as well as the one written by Vandermonde on the knight problem, carried on with the analysis situs initiated by Leibniz. Euler's formula relating the number of edges, vertices, and faes of a convex polyhedron was studied and generalized by Cauchy and L'Huillier, and is at the origin of topology. In particular, the term "graph" was introduced by Sylvester in a paper published in 1878 in Nature, where he draws an analogy between "quantic invariants" and "co-variants" of algebra and molecular diagrams. The first textbook on graph theory was written by Dense Konig, and published in 1936. Another book by Frank Harary, published in 1969.

The origin of graph theory can be traced back to Euler's work on the Konigsberg problem (1736), which subsequently led to the concept of an Eulerian graph. The study of cycles on polyhedra by the Thomas P. Kirkman (1805 – 65) led to the concept of a Hamiltonian graph.

The concept of a tree, a connected graph without cycles, appeared implicity in the work of Gustav Kirchhoff (1824 – 87), who employed graph – theoretical ideas in the calculation of currents in electrical networks or circuits. Later, Arthur Cayley (1821- 95), James J. Sylvester(1806-97), George Polya(1887-1985), and others use 'tree' to enumerate chemical molecules. The study of planar graphs originated in two recreational problems involving the complete graph K_5 and the bipartite graph $K_{3,3}$. These graphs proved to be planarity, as was subsequently demonstrated by Kuratowski.

2. Preliminaries

Let G be a finite, simple, undirected (p,q) graph with vertex set V(G) and edge set E(G).

Definition 2.1

A graph G consists of a pair (V(G), E(G)) where V(G) is a non-empty finite set whose elements are called points or vertices and E(G) is a set of unordered pairs of distinct elements of V(G). The elements of E(G) are called lines or edges of the graph G.

Definition 2.2

A graph H = (V_1, E_1) is called a **subgraph** of G = (V, E) if $V_1 \subseteq V$ and $E_1 \subseteq E$. If H is a subgraph of G we say that G is a **supergraph** of H. H is called a **spanning subgraph** of G if V_1 = V. H is called an **induced subgraph** of G if H is the maximal subgraph of G with point set V_1. Thus, if H is an induced subgraph of G, two points are adjacent in H if and only if they are adjacent in G.

Definition 2.3

A graph in which every two distinct points are adjacent is called a **complete graph**. The complete graph with p points is denoted by $\mathbf{K_p}$.

Definition 2.4

A graph G is called a **bigraph** or **bipartite graph** if V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joins a point of V_1 to a point of V_2. (V_1, V_2) is called a **bipartition** of G. If further G contains every line joining the points of V_1 to the points of V_2 then G is called a **complete bigraph**. If V_1 contains *m* points and V_2 contains *n* points then the complete bigraph G is denoted by $\mathbf{K}_{m,n}$.. $\mathbf{K}_{1,n}$ is called a **star graph** for $n \geq 1$.

Definition 2.5

A **walk** of a graph G is an alternating sequence of points and lines $v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n$ beginning and ending with points such that each line e_i is incident with v_{i-1} and v_i. We say that the walks join v_0 and v_n and it is called a $v_0 - v_n$ walk. v_0 is called the **initial point** and v_n s called the **terminal point** of the walk. The above walk is also denoted by $v_0, v_1, v_2, \ldots, v_n$ the lines of the walk being self evident. A walk is called a **path** if all its points are distinct. A path with n vertices is denoted by $\mathbf{P_n}$.

Definition 2.6

A $v_0 - v_n$ walk is closed if $v_0 = v_n$. A closed walk $v_0, v_1, v_2, \ldots v_n = v_0$ in which $n \geq 3$ and $v_0, v_1, v_2, \ldots, v_{n-1}$ are distinct is called a **cycle** with n-1 vertices. The cycle consisting n vertices is denoted by $\mathbf{C_n}$, the number of lines in the cycle is known as **length** of that cycle. The length of C_n is n.

Definition 2.7

A graph that contains no cycles is called an **acyclic graph**. A connected acyclic graph is called a **tree**.

Definition 2.8

The **degree** of a vertex v_i in a graph G is the number of lines incident with vi. The degree of v_i is denoted by $\mathbf{d_G(v_i)}$ or **deg** $\mathbf{v_i}$ or simply $\mathbf{d(v_i)}$. A vertex v of degree 1 is called a **pendent vertex** or **end vertex**. For any graph G, we define $\delta(G) = \min\{\deg v / v \in V(G)\}$ and $\Delta(G) = \max\{\deg v / v \in V(G)\}$. If all the points of G have the same degree r then $\delta(G) = \Delta(G) = r$ and in this case G is called a **regular graph** of degree r.

Definition 2.9

Let G be a connected graph and $v \in V(G)$. The **eccentricity e(v)** of v is the distance to a vertex farthest from v. Thus $e(v) = \max\{d(u,v) : u \in V(G)\}$. The **radius r(G)** is the minimum eccentricity of the vertices, whereas the **diameter diam(G)** is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. The vertex v is a central vertex if $e(v) = r(G)$. The **center C(G)** is the set of all central vertices. The **central subgraph <C(G)>** of the graph G is the subgraph induced by the center. For a vertex v, each vertex at a distance e(v) from v is an **eccentric vertex of v.** The **Eccentric set of a vertex v** is defined as $E(v) = \{ u \in V(G) / d(u,v) = e(v)\}$.

Definition 2.10

The **open neighborhood N(v)** of a vertex v is the set of all vertices adjacent to v in G. $N[v] = N(v) + \{v\}$ is known as the **closed neighborhood** of v.

For a vertex $v \in V(G)$, $N_i(v) = \{u \in V(G): d(u,v) = i\}$ is defined to be the $\mathbf{i^{th}}$ **neighborhood** of v in G.

Definition 2.11

A set $D \subseteq V(G)$ is a **dominating set** of G, if every vertex in V-D is adjacent to some vertex in D. The dominating set D is a **minimal dominating set** if no proper subset D′ of D is a dominating set. The minimal dominating set with minimum cardinality is known as a **minimum dominating set**. The cardinality of minimum dominating set is known as the **domination number** and is denoted by $\boldsymbol{\gamma(G)}$.

Definition 2.12

A dominating set $D \subseteq V(G)$ is a **connected dominating set** if the induced subgraph <D> is connected. The connected dominating set D is **minimal connected dominating set** if no proper subset D′ of D is a connected dominating set. The minimal connected dominating set with minimum cardinality is known as a **minimum connected dominating set**. The cardinality of minimum connected dominating set is known as a **connected domination number** and is denoted by $\boldsymbol{\gamma_c(G)}$.

Definition 2.13

A set $S \subseteq V(G)$ is known as an **eccentric point set** of G if for every $v \in V\text{-}S$ there exist at least one vertex u in S such that $u \in E(v)$. An eccentric point set S of G is a **minimal eccentric point set** if no proper subset S′ of S is an eccentric point set of G. A minimal eccentric point set with minimum cardinality is known as **minimum eccentric point set** of G. The cardinality of the minimum eccentric point set of G is called as an **eccentric number** of G and is denoted by **e(G)**.

Definition 2.14

A set $D \subseteq V(G)$ is an **eccentric dominating set** if D is a dominating set of G and also for every v in V-D there exist at least one eccentric point of v in D. The eccentric dominating set is a **minimal eccentric dominating set** if no proper subset D′ of D is an eccentric dominating set. The minimal eccentric dominating set with minimum cardinality is called as a **minimum eccentric dominating set**. The cardinality of minimum eccentric dominating set is known as **eccentric domination number** and is denoted by γ_{ed} **(G)**.

Definition 2.15

A set $D \subseteq V(G)$ is a **total eccentric dominating set** if D is an eccentric dominating set of G and also the induced subgraph <D> has no isolated vertices. The total eccentric dominating set is a **minimal total eccentric dominating set** if no proper subset D′ of D is a total eccentric dominating set. The minimal total eccentric dominating set with minimum cardinality is called as a **minimum total eccentric dominating set**. The cardinality of minimum total eccentric dominating set is called as the **total eccentric domination number** and is denoted by γ_{te} **(G).**

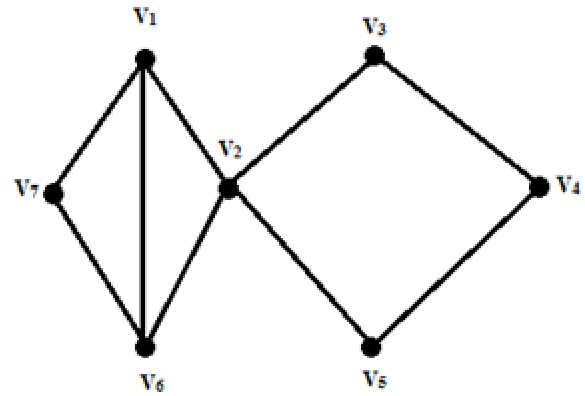
3. Connected Eccentric Domination

A set $D \subseteq V(G)$ is a **connected eccentric dominating set** if D is an eccentric dominating set of G and also the induced subgraph <D> is connected. The connected eccentric dominating set is a **minimal connected eccentric dominating set** if no proper subset D′ of D is a connected eccentric dominating set. The minimal connected eccentric dominating set with minimum cardinality is known as a **minimum connected eccentric dominating set**. The cardinality of minimum connected eccentric dominating set is known as the **connected eccentric domination number** and is denoted by γ_{ce} **(G)**.

Observations

1. It is easy to observe that only connected graphs have a connected eccentric dominating set.
2. Every connected eccentric dominating set is the eccentric dominating set and every eccentric dominating set is the dominating set. Therefore we have $\gamma(G) \leq \gamma_{ed}(G) \leq \gamma_{ce}(G)$.
3. Every connected eccentric dominating set is the connected dominating set and every connected dominating set is the dominating set. Therefore, we have $\gamma(G) \leq \gamma_c(G) \leq \gamma_{ce}(G)$.

Example



$D=\{V_1, V_4\}$

$D_1 = \{V_1, V_2, V_3, V_4, V_7\}$ $D_2 = \{V_2, V_4, V_7\}$

$E(V_1) = \{V_4\}$ $E(V_2) = \{V_4, V_7\}$

$E(V_3) = \{V_7\}$ $E(V_4) = \{V_7\}$

$E(V_5) = \{V_7\}$ $E(V_6) = \{V_4\}$

$E(V_7) = \{V_4\}$

D is a dominating set but not eccentric dominating set. D_2 is a eccentric dominating set but not connected eccentric dominating set. D_1 is a minimum connected eccentric dominating set. Therefore, $\gamma_{ce}(G) = 5$.

Note:

Every non-trivial connected eccentric dominating set is a total eccentric dominating set.

Theorem 3.1

$$\gamma_{ce}(K_n) = 1$$

Proof:

When $G = K_n$, radius r = diameter diam = 1.

Since each vertex $u \in V(G)$ is adjacent to remaining vertices of V(G) and also each vertex $u \in V(G)$ is an eccentric vertex of remaining vertices of V(G).

Hence any vertex $u \in V(G)$ dominates remaining vertices of V(G) and it is also eccentric vertex of remaining vertices and also it is evident that every trivial graph is connected. So D = {u} is a minimum connected eccentric dominating set of G. Therefore, $\gamma_{ce}(K_n) = 1$ when $G = K_n$.

Theorem 3.2:

$$\gamma_{ce}(K_{1,n}) = 2,\ n \geq 2$$

Proof:

When $G = K_{1,n}$ radius r = 1, diameter diam = 2.

Let S = {u, v} where v is a central vertex. The central vertex v dominates all other vertices in V-S and u is an eccentric vertex of vertices of V-S. The induced subgraph <S> is connected.

Therefore, $\gamma_{ce}(K_{1,n}) = 2$, $n \geq 2$.

Theorem 3.3:

$$\gamma_{ce}(K_{m,n}) = 1 \text{ if } m = n = 1$$

$$\gamma_{ce}(K_{m,n}) = 2, \text{ if either } m \geq 1, n \geq 2 \text{ or } m \geq 2, n \geq 1$$

Proof :

If m = n =1 then $K_{1,1} = K_2$. By using theorem 3.1 $\gamma_{ce}(K_{m,n}) = 1$ if m = n = 1.

Consider the case $m \geq 1, n \geq 2$ or $m \geq 2, n \geq 1$. If $m = 1, n \geq 2$ and $m \geq 2$, n = 1 then radius =1, diameter =2 and for $m \geq 2$, $n \geq 2$ we have radius r = diameter diam = 2. Take $G = K_{m,n}$ where $V(G) = V_1 \cup V_2$, $|V_1| = m$ and $|V_2| = n$ and such that each element of V_1 is adjacent to every vertex of V_2 and viceversa.

Let D = {u, v}, $u \in V_1$ and $v \in V_2$, u dominates all the vertices of V_2 and it is an eccentric vertex of all vertices of V_1-{u}. Similarly v dominates all the vertices of V_1 and it is an eccentric vertex of all vertices of V_2-{v}. The induced subgraph <D> is connected. Therefore, $\gamma_{ce}(K_{m,n}) = 2$, if either $m \geq 1, n \geq 2$ or $m \geq 2, n \geq 1$.

Theorem 3.4:

$$\gamma_{ce}(C_n) = n-2,\ n \geq 3$$

Proof :

When $G = C_n$, radius $r = \begin{cases} n/2 \text{ when n is even} \\ (n-1)/2 \text{ when n is odd} \end{cases}$

In C_n, radius r = diameter diam. Consider the cycle $C_n : V_1, V_2, V_3,\ldots\ldots,V_n,V_{n+1} = V_1$. Since in C_n every vertex is 2 – regular, each vertex of $V(C_n)$ dominates exactly 3 vertices. The vertex V_1 dominates V_2, V_n and itself. Now include the vertex V_1 in the set D. In order to form a connected dominating set D, We have to include next consecutive vertex either V_2 or V_n in D, otherwise we can't form a connected dominating set. Suppose we select $V_2 \in D$, then we have to choose next consecutive vertex V_3 in D. This process is continued until we have (n-2) vertices of C_n in D. Therefore, $D = \{ V_1, V_2, V_3,\ldots\ldots,V_{n-2}\}$. The vertices $V_{n-1} \in V-D$ is dominated by V_{n-2} of D and the vertex $V_n \in V-D$ is dominated by $V_1 \in D$. Clearly D is the minimum connected dominating set of C_n. We know that C_n is a self – centered graph and radius = r.

The eccentric vertex of V_i = V_{i+r} if $i \leq r$

V_{i-r} if $i > r$ when n is even and odd

When n is odd there exist another one eccentric vertex of

$$V_i = \begin{cases} V_{i+r+1} \text{ if } i \leq r \\ V_{i-r-1} \text{ if } i > r+1 \\ V_n \text{ if } i = r+1 \end{cases}$$

Case (i): When n is even.

Here r = n/2.

$V_{n-1} \in V-D$.

The eccentric vertex o f $V_{n-1} = V_{n-1-(n/2)}$, since $n-1 > (n/2)$

$= V_{(2n-2-n)/2}$

$= V_{(n-2)/2} \neq V_n$

Therefore the eccentric vertex of $V_{n-1} = V_{(n-2)/2} \in D$

$V_n \in V-D$.

The eccentric vertex of $V_n = V_{n-(n/2)}$, since $n > (n/2)$

$= V_{n/2} \neq V_{n-1}$

Therefore the eccentric vertex of $V_n = V_{n/2} \in D$

Case (ii) : When n is odd

Here r = (n-1)/2 and r+1 = (n+1)/2.

$V_{n-1} \in V-D$.

The eccentric vertices of $V_{n-1} = \{V_{n-1-((n-1)/2)}, V_{n-1-((n-1)/2)-1}\}$

[since n-1 > ((n-1)/2) and n-1 > r+1]

= { $V_{(2n-2-n+1)/2}$, $V_{(2n-2-n+1-2)/2}$ }

= {$V_{(n-1)/2}$, $V_{(n-3)/2}$} ∈ D for n > 3

[since $V_{(n-1)/2} \neq V_n \neq V_{(n-3)/2}$]

= {$V_{(n-1)/2}$} ∈ D for n = 3

[since $V_{(n-1)/2} \neq V_n = V_{(n-3)/2}$ put $V_0 = V_n$]

$V_n \in$ V-D.

The eccentric vertices of V_n = {$V_{n-((n-1)/2)}$, $V_{n-((n-1)/2)-1}$}
[since n-1 > ((n-1)/2) and n-1 > r+1]

= { $V_{(2n-n+1)/2}$, $V_{(2n-n+1-2)/2}$ }
= {$V_{(n+1)/2}$, $V_{(n-1)/2}$} ∈ D for n > 3
[since $V_{(n+1)/2} \neq V_{n-1} \neq V_{(n-1)/2}$]
= {$V_{(n-1)/2}$} ∈ D for n = 3
[since $V_{(n-1)/2} \neq V_{n-1} = V_{(n+1)/2}$]

In both cases the eccentric vertex of V_{n-1} and V_n is in D and the vertex V_{n-1} is dominated by the vertex V_{n-2} of D and the vertex V_n is dominated by the vertex V_1 of D and also D is a minimum connected dominating set of C_n. Then clearly D is the minimum connected eccentric dominating set. Therefore, $\gamma_{ce}(C_n)$ = |D| = n-2. Hence $\gamma_{ce}(C_n)$ = n-2, n ≥ 3.

Theorem 3.5:

$\gamma_{ce}(W_3) = 1$

$\gamma_{ce}(W_4) = 2$

$\gamma_{ce}(W_n) = 3, \quad n \geq 5$

Proof:

If G = W_3 = K_4. Therefore by theorem 3.1, $\gamma_{ce}(K_4)$ = 1 which implies that $\gamma_{ce}(W_3)$ = 1

When G = W_4, consider D = {u, v} where u and v are adjacent non central vertices. D is a minimum connected eccentric dominating set. Therefore, $\gamma_{ce}(W_4)$ = 2.

When G = W_n, n ≥ 5, consider D = {u, v, w} where v is central vertex and u, w are any two adjacent non-central vertices. D is a minimum connected eccentric dominating set. Therefore, $\gamma_{ce}(W_n)$ = 3, n ≥ 5.

Theorem 3.6:

If G is of diameter two then $\gamma_{ce}(G) \leq 1 + \delta(G)$.

Proof :

diam(G) = 2. Let w ∈ V(G) such that deg w = δ(G). Consider D = {w} ∪ N(w). This is a connected eccentric dominating set of G. The induced subgraph <D> is connected. Therefore, $\gamma_{ce}(G) \leq 1 + \delta(G)$.

Theorem 3.7:

If the tree T is of radius 2 with unique central vertex u and deg v ≤ 2 for every v ∈ N(u) then
$\gamma_{ce}(G) \leq$ deg (u) + 2.

Proof :

Let the tree T is of radius 2 with unique central vertex u. Then N[u] is a connected dominating set for G.

Case (i):

If any vertex v ∈ N(u) is a pendent vertex then N[u] – {v} is a minimum connected dominating set. Suppose if there are k pendent vertex in N(u), put all that vertex in the set S. Then N[u] – {S} is the minimum connected dominating set for G. Any vertex w in V – N[u] is an eccentric vertex for all other remaining vertices V – N[u] and also for the vertices of S.

Therefore, N[u] - {S} + {w} is a minimum connected eccentric dominating set.

$\gamma_{ce}(G)$ = |N[u] –{S} + {w}|

= deg(u) + 1 – k + 1

= deg(u) + 2 – k

< deg(u) + 2

Case (ii) :

If no vertex of N(u) is a pendent vertex then N[u] is the minimum connected dominating set. Any vertex w ∈ V – N[u] is an eccentric vertex for all other vertices of V – N[u]. Therefore, N[u] + {w} is minimum connected eccentric dominating set for G.

$\gamma_{ce}(G)$ = |N[u] + {w}|

= deg(u) + 1 + 1

= deg(u) + 2

Hence from case (i) and case (ii) $\gamma_{ce}(G) \leq deg(u) + 2$ where u is a central vertex which is unique and of radius 2.

Theorem 3.8:

$$\gamma_{ce}(P_n) = n-1$$

Proof:

Let P_n be a path with n vertices. Then there exist 2 pendent vertices say u, v in P_n. Evidently $D = V(P_n) - \{u, v\}$ is the minimum connected dominating set. But the eccentric vertex of $u \in V(P_n) - D$ is $v \in V(P_n) - D$ and the eccentric vertex of $v \in V(P_n) - D$ is $u \in V(P_n) - D$, therefore we have to add either u or v in D to form the eccentric dominating set. So that take $D = V(P_n) - \{v\}$, then clearly D is the minimum connected eccentric dominating set and $|D| = n-1$.

Hence $\gamma_{ce}(P_n) = n-1$.

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